Chapter 4

Simultaneous equations

Essential reading

See Chapter 4 of *Dowling* for many further examples of the material covered in this chapter. In particular, use the supplementary problems 4.31 to 4.40 to test your understanding of systems of equations and their applications.

Some practical problems require several equations and variables to describe or model them in mathematical terms. A solution must satisfy all of the equations at the same time and hence such equations are known as *simultaneous equations*.

Simultaneous equations can be solved using graphs or using algebra. We will be looking at both methods of solution in this chapter. We will concentrate on solving two equations with two variables. This is known as a $2 * 2$ *system of equations*.

Systems of simultaneous equations occur naturally in business and economics. For example, one equation may represent supply and another demand. Solving these equations simultaneously will tell us when supply and demand are balanced, bringing equilibrium to the market. As well as supply and demand analysis, we will also study income determination, and IS-LM analysis which links national income and interest rates with the money market.

4.1 Systems of equations

We know how to solve an equation with one variable using algebraic methods such as changing the subject or quadratic factorisation.

However, if we have an equation with two unknown variables such as:

$$2y = 4x + 6$$

then we cannot find a unique solution. We can only say what y will equal if we give x a particular value. For example if $x = 4$ then $y = 11$ and if $x = 9$ then $y = 21$. There are *infinitely many solutions* because x can take infinitely many values and whatever value we give x there will be a corresponding solution for y.

In order to find a unique solution for two variables, we need two equations.

For example, consider the following equations:

$$8 = x + 2y$$
$$11 = x + 3y$$

Here we have two equations each with two variables x and y. This is called a $2 * 2$ system of equations. We want to find a solution which *satisfies* both of the equations i.e., we want to find a value for x and a value for y so that both of the equations are correct.

The second equation has the same number of x's but one more y than the first equation. Therefore the difference between the left-hand-side of the first and second equation must equal one y. $11 - 8 = 3$ so we must have $y = 3$. Now we know $y = 3$ we can substitute this into the first equation to make one equation in one variable:

$$8 = x + 2(3)$$

Now we can solve this equation to find $x = 2$.

We have used an algebraic method known as *elimination* to find the solution, $x = 2$ and $y = 3$, for this system of equations. We will look at the elimination method in more detail later on.

4.1.1 Unique, multiple, infinite and impossible solutions

A set of n equations with m variables is called an $n * m$ *system of equations*.

Unique solutions

The solution $x = 2, y = 3$ to the $2 * 2$ system of equations in the example above is *unique*. No other values of x and y can make both equations correct at the same time.

In order to have a unique solution, a system of equations must have at least as many equations as variables. Therefore we can usually find a unique solution to an $n * m$ system of equations - that is a system of n equations with m variables - if $n \geq m$. Note however that there are some exceptions to this - in particular if the equations are non-distinct (see section 4.2.1).

Infinite solutions

We saw earlier that if we have one equation with two variables then we cannot find a unique solution. In our example, $2y = 4x + 6$, there are infinitely many solutions.

It is true in general that if we have more variables than equations i.e., $n < m$ in an $n * m$ system, then we will not be able to find a unique solution. For example, below is a $2 * 3$ system of equations which has only two equations but three variables:

$$\begin{aligned} 10 &= x + y + z \\ -4 &= x - y - z \end{aligned}$$

We can find a solution to this system of equations:
$x = 3, y = 5, z = 2$ is a solution.

However, so is $x = 3, y = 3, z = 4$ or $x = 3, y = 1, z = 6$.

The value of x must be 3, but we can pick any values for y and z so long as $y + z = 7$. (See section 4.3.1 to understand why this is so.)

This means that there are infinitely many solutions because we can assign whatever value we like to y and then make $z = 7 - y$. For example, $x = 3$, $y = 100$, $z = -93$.

Multiple solutions

If one of the equations in a system is a quadratic then there may be two correct solutions. For example consider the following $2 * 2$ system:

$$y = x^2$$
$$y = 5x - 6$$

We can solve this system by *equating* the two equations. This means that we set the two right-hand-sides equal to each other to make a quadratic equation in one variable x which we can solve to find two solutions for x. Then by substituting these values for x into one of the original equations, we can find the corresponding values of y.

$$x^2 = 5x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$

Either $x = 2$ or $x = 3$. When $x = 2$, $y = 2^2$ and when $x = 3$, $y = 3^2$ therefore this system has two possible solutions. The first is $x = 2, y = 4$ and the second is $x = 3, y = 9$.

No solutions

Consider the following $2 * 2$ system:

$$4 = y - x$$
$$6 = y - x$$

There are no values which can be assigned to x and y which will make both of these equations correct. Try giving x a value and then making $y - x$ equal both 4 and 6 at the same time - it's impossible. Therefore this system of equations has no solution.

4.2 Using graphs to solve simultaneous equations

It is much easier to understand why some simultaneous equations have no solution and why others have multiple solutions when we consider the equations graphically.

As we have seen in chapters 2 and 3, we can draw the graph of an equation. If the equation is linear it will take the form $y = mx + c$

and will be a straight line graph. If the equation is quadratic then it will include a square term and the graph will be a parabola.

We can solve a system of simultaneous equations by drawing the line of each equation on the same graph. The point(s) where the lines cross are the solutions to the equations - read the x value off the x axis and the y value off the y axis.

4.2.1 Systems of linear equations

If we draw the lines of two linear equations (straight lines) on the same graph then they might cross:

- never - if the two lines are parallel - no solution;
- once - if the two lines are different and not parallel - unique solution;
- infinitely - if the two lines are actually the same line - infinite solutions.

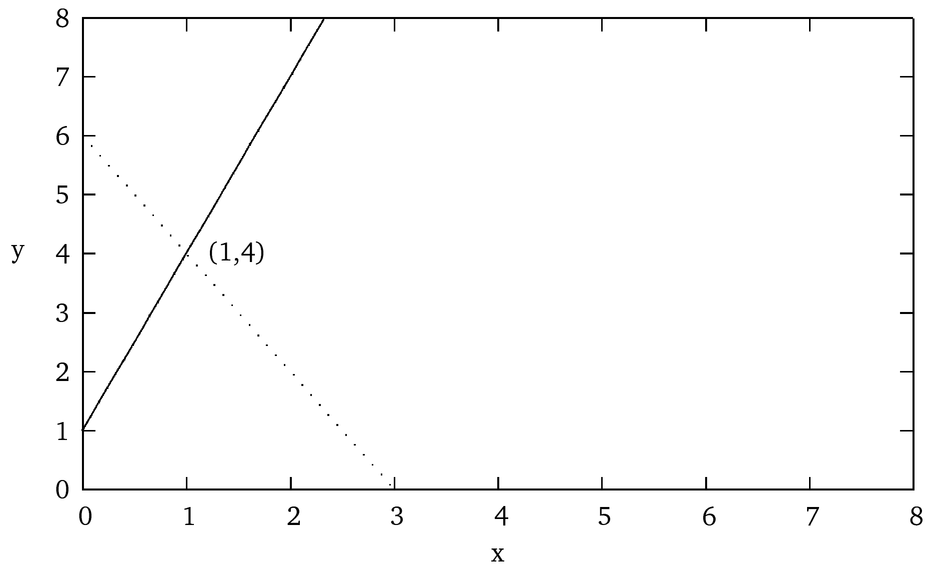
Independent equations : unique solution

Two linear equations are said to be *independent* if their lines are distinct and not parallel. Two independent lines will cross each other at exactly one point and at this point both equations are solved simultaneously.

For example consider the system of linear equations below:

$$y = 3x + 1$$
$$y = -2x + 6$$

Lines representing each of these equations are drawn on the graph below:

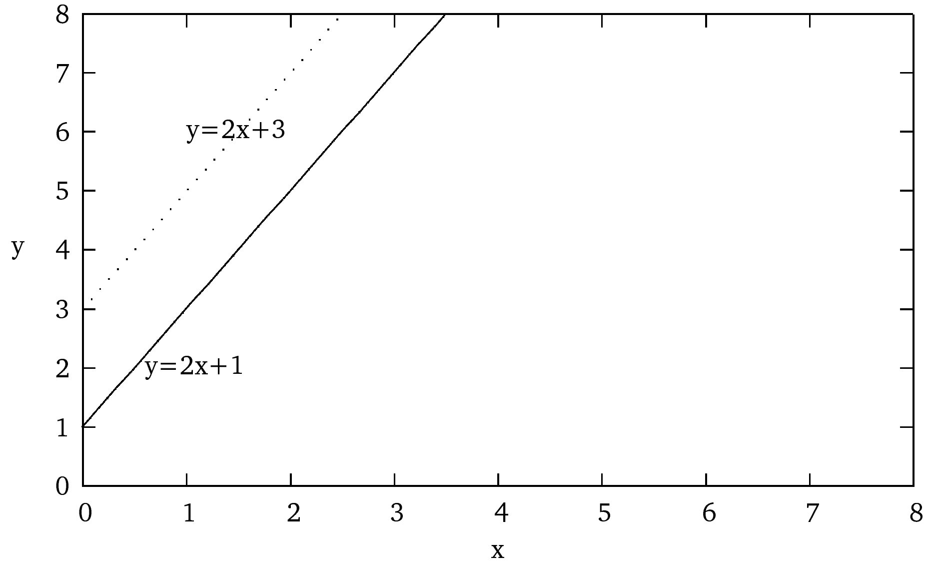


The point where the two lines cross is $(1, 4)$ and the solution is $x = 1, y = 4$.

Parallel lines : no solution

If two straight lines have the same gradient then they will be parallel to each other and they will never cross.

The graph below shows the two lines $y = 2x + 1$ and $y = 2x + 3$.



These two lines both have gradient 2 and are parallel. They will never cross each other. This explains why the system of equations below has no solution.

$$y = 2x + 1$$
$$y = 2x + 3$$

Non-independent lines : infinitely many solutions

Consider the two linear equations:

$$4 = x + 3y$$
$$12 = 3x + 9y$$

If you plot these two lines on the same graph, you will end up with just one line. This is because the terms in the second equation are all 3 times the terms in the first equation. The second equation is a *multiple* of the first.

Such equations are said to be *non-independent* and any solution of the first equation will also be a solution for the second.

Since in effect we have just one equation but two variables we cannot find a unique solution to a system with non-independent equations.

In our example the first equation has infinitely many solutions (since we can pick any value for x and then compute $y = (4 - x)/3$), and therefore the system of equations also has infinitely many solutions.

Learning activity

State whether the following systems of equations have no solution, a unique solution or infinitely many solutions. For those systems with a unique solution find this solution by sketching a graph of the two lines and finding their meeting point.

1.

$$\begin{aligned} y &= 2x + 3 \\ y &= 3x - 1 \end{aligned}$$

2.

$$\begin{aligned} y + 2x &= 3 \\ -2y - 8x &= -6 \end{aligned}$$

3.

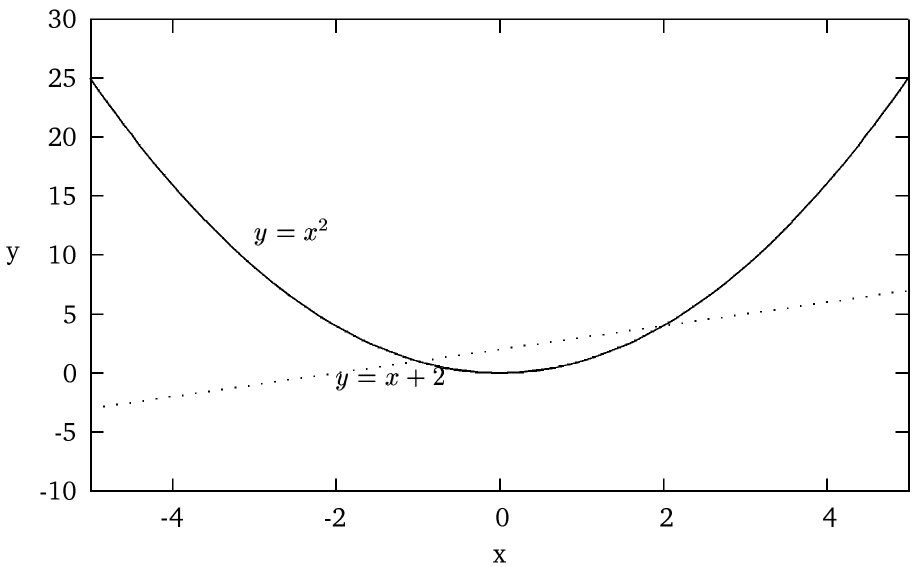
$$\begin{aligned} 2y - 4x - 2 &= 0 \\ y - 2x + 1 &= 0 \end{aligned}$$

4.

$$\begin{aligned} y &= -5x + 8 \\ y &= 5x - 2 \end{aligned}$$

4.2.2 Systems involving quadratic equations

If we draw the graph of a quadratic equation and a straight line then the straight line might cut through the quadratic twice as in the graph below.



The lines in the graph above represent $y = x^2$ and $y = x + 2$. The lines meet at two points $(2, 4)$ and $(-1, 1)$ and therefore the system of simultaneous equations:

$$y = x^2$$
$$y = x + 2$$

has two solutions. Namely $x = 2, y = 4$ and $x = -1, y = 1$.

Learning activity

As shown in the graph above, a straight line may cut a quadratic at two points. It is also possible for a straight line to cut a quadratic just once or not at all. Draw graphs which show:

1. A quadratic and a straight line which meet at exactly one point.
2. A quadratic and a straight line which do not meet at all.

4.3 Using algebra to solve simultaneous equations

Although it can sometimes be helpful to draw a graph when solving simultaneous equations, it is not always necessary. We can use algebraic methods of solution instead. There are two different techniques which we can use - *elimination* or *substitution*. When using either technique the aim is the same. If we start off with two equations in two unknowns, we try to get rid of one of the variables to make one equation in one unknown.

4.3.1 Solution by elimination

Suppose we want to solve a $2 * 2$ system of linear equations. Each equation has 2 variables x and y say. The method of elimination combines the two equations, by adding or subtracting them term by term, to make one equation in one variable. This equation can then be solved to find the value of the one of the variables. The value of the second variable can then be found by substituting the value of the known variable into one of the original equations.

Example 1

Consider the following system of equations:

$$3y + 2x = -5 \tag{4.1}$$
$$y + 2x = 1 \tag{4.2}$$

Both of the equations include the term $+2x$. If we subtract the second equation from the first term by term then the $+2x$ terms are eliminated because they cancel each other out. We are left with:

$$2y = -6$$

This equation can be solved for y and gives us $y = -3$.

By substituting $y = -3$ back into equation 4.1 we can find the value of x.

$$\begin{aligned} 3(-3) + 2x &= -5 \\ 2x &= -5 + 9 \\ x &= 4/2 = 2 \end{aligned}$$

We have found the solution $x = 2, y = -3$. We can check this answer by making sure that it also satisfies equation 4.2.

$$(-3) + 2(2) = -3 + 4 = 1$$

Our solution $x = 2, y = -3$ satisfies both of the equations 4.1 and 4.2 and therefore we can be sure that it is the correct solution.

Example 2

In the previous example, it was easy to eliminate the x term because exactly the same term $+2x$ appeared in both of the equations. If the equations do not both include the same term then we must manipulate one or both of the equations to make elimination possible. For example, consider the following system of equations:

$$2y + 3x = 5 \qquad (4.3)$$

$$3y + 6x = 6 \qquad (4.4)$$

This time we do not have a pair of matching terms. However, if we multiply all of the terms in equation 4.3 by 2 then we get a new equation:

$$4y + 6x = 10 \qquad (4.5)$$

Equations 4.3 and 4.5 are non-independent and will have the same solutions. We can solve the system of equations 4.4 and 4.5 by subtracting one from the other to eliminate the $+6x$ terms.

$$\begin{aligned} 3y + 6x &= 6 \\ -(\underline{4y + 6x = 10}) \\ -y &= -4 \end{aligned}$$

We have found that $y = 4$ and by substituting this value for y into equation 4.3 we can find the value of x.

$$\begin{aligned} 2(4) + 3x &= 5 \\ 3x &= 5 - 8 \\ x &= -3/3 = -1 \end{aligned}$$

Our solution is $x = -1, y = 4$. We can check this answer by substituting these values for x and y into equation 4.4:

$$3y + 6x = 3(4) + 6(-1) = 12 - 6 = 6$$

Both equations 4.3 and 4.4 are satisfied by the solution $x = -1, y = 4$.

Example 3

In both of the previous examples, we have subtracted one of the equations from the other in order to eliminate one of the variables. In the following example, we will add the equations together because the term $3x$ that we want to eliminate is positive in the first equation and negative in the second equation.

$$4y + 3x = 16 \tag{4.6}$$
$$5y - 3x = 47 \tag{4.7}$$

Adding equations 4.6 and 4.7 together term by term gives:

$$9y = 63$$

We can solve this equation to find $y = 7$. Substituting $y = 7$ into equation 4.6 will tell us the value of x:

$$\begin{aligned} 4(7) + 3x &= 16 \\ 3x &= 16 - 28 \\ x &= -12/3 = -4 \end{aligned}$$

We check the solution $x = -4, y = 7$ by substituting these values in equation 4.7:

$$5(7) - 3(-4) = 35 + 12 = 47$$

Since both equations are satisfied we are sure that the solution $x = -4, y = 7$ is correct.

Example 4

What if there are no terms in the equations which can cancel with each other? Sometimes we have to multiply both of the equations by a constant so that each will contain a cancelling term. This is the case in the next example.

$$2y + 5x = 34 \tag{4.8}$$
$$3y + 2x = 7 \tag{4.9}$$

Neither the x nor the y terms will cancel if we add or subtract the equations. To remedy this, we will multiply equation 4.8 by 3 and equation 4.9 by 2. This will make the term $6y$ appear in both equations.

$$6y + 15x = 102 \tag{4.10}$$
$$6y + 4x = 14 \tag{4.11}$$

Note that equation 4.10 has the same solutions as equation 4.8 and equation 4.11 has the same solutions as equation 4.9. Therefore we can solve the original system of equations by solving this second system of equations.

This can be done by subtracting equation 4.11 from equation 4.10 to obtain:

$$\begin{aligned} 11x &= 88 \\ x &= 88/11 = 8 \end{aligned}$$

Now substituting $x = 8$ into equation 4.8 gives us:

$$\begin{aligned} 2y + 5(8) &= 34 \\ 2y &= 34 - 40 \\ y &= -6/2 = -3 \end{aligned}$$

Finally we check our solution by substituting the values $x = 8$, $y = -3$ into equation 4.9:

$$3(-3) + 2(8) = -9 + 16 = 7$$

Both of the equations 4.8 and 4.9 are satisfied and so we can be sure that the solution $x = 8, y = -3$ is correct.

Learning activity

Solve the following simultaneous equations using the method of elimination.

1.

$$\begin{aligned} x + 2y &= 9 \\ 3x + y &= 7 \end{aligned}$$

2.

$$\begin{aligned} 7x + 3y &= 27 \\ 2x - y &= 4 \end{aligned}$$

3.

$$\begin{aligned} 2x + 3y &= 27 \\ 3x + 2y &= 28 \end{aligned}$$

4.

$$\begin{aligned} 2x + 5y &= 16 \\ 5x + 3y &= 21 \end{aligned}$$

4.3.2 Solution by substitution

An alternative algebraic method is to rearrange one of the equations so that one of the variables is expressed in terms of the other. Substituting this into the second equation results in one equation in one variable. The following examples show how the method works.

Example 5

$$2x + 3y = 14 \quad (4.12)$$
$$3x - y = 10 \quad (4.13)$$

We can rearrange equation 4.13 to make y the subject:

$$y = 3x - 10$$

Now substituting $y = 3x - 10$ into equation 4.12 gives:

$$\begin{aligned} 2x + 3(3x - 10) &= 14 \\ 2x + 9x - 30 &= 14 \\ 11x &= 44 \\ x &= 4 \end{aligned}$$

We have found that $x = 4$, substituting this into equation 4.12 tells us the value of y:

$$\begin{aligned} 2(4) + 3y &= 14 \\ 3y &= 14 - 8 \\ y &= 6/3 = 2 \end{aligned}$$

We can check the solution by substituting $x = 4, y = 2$ into equation 4.13:

$$3(4) - (2) = 10$$

Example 6

The following example involves a quadratic equation and a linear equation.

$$y = 11x - 2 \quad (4.14)$$
$$y = 5x^2 \quad (4.15)$$

This time there is no need to rearrange either of the equations. We simply substitute $y = 5x^2$ into equation 4.14 to obtain a quadratic equation in one variable:

$$\begin{aligned} 5x^2 &= 11x - 2 \\ 5x^2 - 11x + 2 &= 0 \end{aligned}$$

Now we can use the quadratic formula (3.3.1) with $a = 5$, $b = -11$ and $c = 2$ to find the values of x which satisfy this quadratic:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$\begin{aligned}
x &= \frac{11 \pm \sqrt{((-11)^2 - 4(5)(2))}}{2(5)} \\
x &= \frac{11 \pm 9}{10} \\
x &= 2 \quad or \quad x = 0.2
\end{aligned}$$

We now substitute the solutions $x = 2$ and $x = 0.2$ into equation 4.14 in turn to find the corresponding solutions for y:

When $x = 2$ we have $y = 11(2) - 2 = 20$.

When $x = 0.2$ we have $y = 11(0.2) - 2 = 0.2$.

We can check both of these solutions by substituting the values of x and y into equation 4.15:

$$5x^2 = 5(0.2)^2 = 0.2 = y$$
$$5x^2 = 5(2)^2 = 20 = y$$

Equation 4.15 is satisfied and therefore both pairs of solutions $x = 0.2, y = 0.2$ and $x = 4, y = 20$ are correct.

Example 7

$$x + y = 7 \tag{4.16}$$
$$x^2 + 2y^2 = 54 \tag{4.17}$$

From equation 4.16 we have $x = y - 7$. Substituting this into equation 4.17 and rearranging gives:

$$\begin{aligned}
(y-7)^2 + 2y^2 &= 54 \\
y^2 - 14y + 49 + 2y^2 &= 54 \\
3y^2 - 14y - 5 &= 0
\end{aligned}$$

Using the quadratic formula with $a = 3$, $b = -14$ and $c = -5$ gives us the solutions $y = -\frac{1}{3}$ and $y = 5$. We substitute these solutions for y into equation 4.16 to find the corresponding solutions for x:

When $y = -\frac{1}{3}$, $x = -7\frac{1}{3}$ and when $y = 5$, $x = 2$.

We can check these solutions by substituting them into equation 4.17:

$$(2)^2 + 2(5)^2 = 4 + 2(25) = 54$$
$$(-7\frac{1}{3})^2 + 2(-\frac{1}{3})^2 = 53\frac{7}{9} + 2(\frac{1}{9}) = 54$$

Sometimes we are given information that we can use to form simultaneous equations as in the following example.

Example 8

In the St James' School Hall, some rows seat 25 people and some seat 15 people. There are 26 rows altogether. When the hall is full it seats 550 people. How many rows seat 25 people and how many seat 15?

We can solve this problem by letting x represent the number of rows with 25 seats and y represent the number of rows with 15 seats and writing the information given as two equations:

$$x + y = 26 \tag{4.18}$$
$$25x + 15y = 550 \tag{4.19}$$

Equation 4.18 represents the information that there are 26 rows of seats altogether. Equation 4.19 uses the information that there are 550 seats altogether.

We can solve this system of equations using substitution. We replace y in equation 4.19 by $(26 - x)$.

$$\begin{aligned} 25x + 15(26 - x) &= 550 \\ 25x + 390 - 15x - 550 &= 0 \\ 10x - 160 &= 0 \\ x &= 16 \end{aligned}$$

We have found that there are 16 rows with 25 seats and this means that there must be 10 rows with 15 seats. We can check this answer by substituting these values for x and y into equation 4.19.

$$25(16) + 15(10) = 400 + 150 = 550$$

Learning activity

1. Solve the following systems of equations using the method of substitution. Give your answers correct to two decimal places where appropriate.

   (a)

   $$\begin{aligned} x + 2y &= 3 \\ x^2 + 2y^2 &= 3 \end{aligned}$$

   (b)

   $$\begin{aligned} x + y &= 5 \\ xy &= 6 \end{aligned}$$

   (c)

   $$\begin{aligned} x + y &= 2 \\ 3x^2 - y^2 &= 1 \end{aligned}$$

2. At a concert tickets cost either \$4 or \$6. A total of 700 tickets are sold at a cost of \$3360. How many \$4 tickets were sold?

3. The difference between two numbers p and q is 21. The same two numbers add to 95. Write down two equations expressing the relationship between p and q. Hence find the values of p and q.
4. The Taylors and the Smiths have booked the same holiday. The Taylor family have to pay \$1880 for two adults and three children. The Smith family will pay \$2110 for three adults and two children. Write down the equation for each family and hence find the cost for each adult and each child.

4.4 Applications in business and economics

4.4.1 Supply and demand analysis

Balancing supply and demand is very important in business - if supply is greater than demand then there will be a surplus of goods and the price will fall. On the other hand if demand is greater than supply then prices will rise but ultimately customers will be unsatisfied. With supply and demand modelled by equations, we can find the point of *equilibrium* by solving the equations simultaneously. At this point, supply and demand are balanced.

Let P represent *price* and Q represent *quantity*. Then equations representing *supply* and *demand* can be written using the two variables P and Q. By solving these equations simultaneously, we can find the *Equilibrium Point* (P_E, Q_E) where supply and demand are equal.

The solution for P will be the *Equilibrium Price* P_E and the solution for Q will be the *Equilibrium Quantity* Q_E.

Worked example

A firm has the following supply and demand equations where Q is the quantity and P is the price of goods produced:

$$\begin{aligned} \text{Supply equation} \quad Q &= -40 + 6P \\ \text{Demand equation} \quad Q &= 240 - 8P \end{aligned}$$

We can find the point of equilibrium by solving the supply and demand equations simultaneously. This can be done graphically or algebraically as described in the previous sections. To demonstrate both methods, we will find the point of equilibrium using the elimination method and then draw the lines on a graph to show that they cross at this point.

Subtracting the supply equation from the demand equation leaves us with:

$$\begin{aligned} 0 &= 280 - 14P \\ 14P &= 280 \\ P &= 280/14 = 20 \end{aligned}$$

This means that the equilibrium price $P_E = 20$.

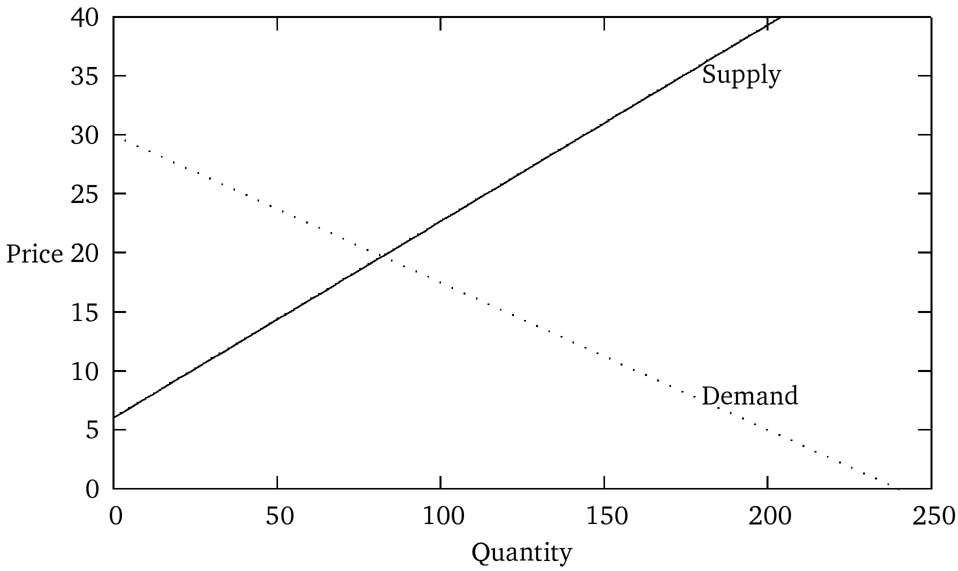
Substituting $P_E = 20$ into the supply equation will tell us the equilibrium quantity Q_E.

$$Q_E = -40 + 6(P_E) = -40 + 6(20) = 80$$

Thus the point of equilibrium is $(20, 80)$.

By drawing the two lines on a graph, we will show that the supply and demand lines cross at the point $(20, 80)$. Since *Quantity* is the controlling variable and *Price* is the dependent variable we should put *Quantity* on the x-axis and *Price* on the y-axis. This means that we need to re-arrange the supply and demand equations to make *Price* the subject:

$$\begin{array}{lll} \text{Supply equation} & 6P & = Q + 40 \\ \text{Demand equation} & 8P & = -Q + 240 \end{array}$$



Looking at the graph, we can see that there is a *surplus* of goods when $Q > 80$ which means that the price falls below the equilibrium price; and a *shortage* when $Q < 80$ which means that the price rises until it reaches the equilibrium price.

4.4.2 Income determination

The economy is in equilibrium if *income* is equal to *expenditure*. In a simple two-sector model, expenditure can be thought of as consumption plus investment. Thus the economy is in equilibrium if

$$income = consumption + investment$$

We usually use the letters:

- I to represent investment;

- C to represent consumption;
- Y to represent income.

When the equation below is satisfied there is income and expenditure balance. This equation is known as the *equilibrium equation*:

$$Y = C + I$$

The *Consumption Function* C is a function of *income* Y. In a simple model, consumption increases as income increases and the relationship between consumption and income can be expressed as:

$$C = C_0 + bY$$

where C_0 and b are constant values.

The amount of investment I can be fixed. This means that we are left with two variables Y and C. The two equations:

$$\begin{aligned} Y &= C + I \\ C &= C_0 + bY \end{aligned}$$

form a $2 * 2$ system which can be solved simultaneously to find the values of Y and C. This is known as *income determination*.

Worked example

For a *closed economy* in which there is no government intervention the consumption function is given by $C = 10 + 0.6Y$ and planned investment is $I = 12$.

Find the equilibrium levels of income and consumption.

Solution We know that $Y = C + I$ and since in this example $I = 12$ we have the following system of equations:

$$\begin{aligned} Y &= C + 12 \\ C &= 10 + 0.6Y \end{aligned}$$

We can solve this system of equations to find the equilibrium income Y by replacing C in the second equation by $Y - 12$ (from the first equation) . We can then solve the resulting equation for Y as follows:

$$\begin{aligned} Y - 12 &= 10 + 0.6Y \\ 0.4Y &= 22 \\ Y &= 22/0.4 = 55 \end{aligned}$$

Now we can find C by substituting $Y = 55$ back into the equilibrium equation:

$$\begin{aligned} Y &= C + 12 \\ 55 &= C + 12 \\ C &= 43 \end{aligned}$$

Thus the equilibrium levels in this model are $income = 55$ and $consumption = 43$.

Government expenditure

It is not very realistic to assume that an economy is closed with no government intervention. To make the model more realistic we can include *government expenditure* G, in the equilibrium equation.

$$Y = C + I + G$$

Given a consumption function and planned values for I and G we can find the equilibrium income by solving simultaneous equations as before.

Worked example

Find the equilibrium level of income given that $C = 135 + 0.8Y$, $I = 75$ and $G = 30$.

Substituting the given values of G and I into equilibrium equation $Y = C + I + G$ gives:

$$\begin{aligned} Y &= C + 75 + 30 \\ Y &= C + 105 \\ C &= Y - 105 \end{aligned}$$

Now by replacing C by $Y - 105$ in the consumption equation $C = 135 + 0.8Y$ we can find the equilibrium value of Y.

$$\begin{aligned} Y - 105 &= 135 + 0.8y \\ 0.2Y &= 240 \\ Y &= 240/0.2 = 1200 \end{aligned}$$

The equilibrium income in this model is $Y = 1200$.

4.4.3 IS-LM analysis

Up until now we have assumed that investment I is a constant value. However it is often more realistic to take I as a function of the rate of interest i. As the rate of interest rises the rate of investment falls. There is a linear relationship between I and i. If we are given the investment function and the consumption function of an economy, we can combine these functions to express the relationship between national income Y and interest rate i.

For example, suppose $C = 100 + 0.8Y$ and $I = -20i + 1000$.

Substituting these values into the equilibrium equation $Y = C + I$ gives:

$$\begin{aligned} Y &= (100 + 0.8Y) + (-20i + 1000) \\ 0.2Y &= 1100 - 20i \end{aligned}$$

This equation relating national income Y to interest rate i is called the **IS schedule**.

We may wish to find the values of Y and i. However we currently have one equation in two variables and so there is no unique solution. The correct solution will occur when the *money market* is in equilibrium. This means that the supply of money M_s is equal to the demand for money M_d.

The demand for money comes from three sources:

- The *transactions demand* — used for the daily exchange of goods and services.
- The *precautionary demand* — used to fund any emergencies requiring unforeseen expenditure.
- The *speculative demand* — used as a reserve fund for investment which falls as interest rates rise.

We use M_t - a linear function in Y, to represent the transactions and precautionary demand, and M_w - a linear function in i to represent the speculative demand. The demand for money $M_d = M_t + M_w$.

Thus the money market is in equilibrium if:

$$M_s = M_t + M_w$$

This equation is called the **LM schedule**.

For example, suppose $M_s = 2375$, $M_t = 0.1Y$ and $M_w = -25i + 2000$. Then the money market is in equilibrium if:

$$2375 = 0.1Y - 25i + 2000$$

Now we have two equations in two variables, namely the IS schedule and the LM schedule. Putting them together, we can solve the equations simultaneously to find the values of i and Y which satisfy the equilibrium equation when the money market is also in equilibrium.

$$\begin{array}{lll} \text{IS schedule} & 0.2Y & = 1100 - 20i \\ \text{LM schedule} & 2375 & = 0.1Y - 25i + 2000 \end{array}$$

Re-arranging the IS schedule gives:

$$Y = 5500 - 100i$$

Now substituting this value for Y in the LM schedule we can find the value of i as follows:

$$\begin{aligned} 2375 &= 0.1(5500 - 100i) - 25i + 2000 \\ 2375 &= 550 - 10i - 25i + 2000 \\ 35i &= 175 \\ i &= 5 \end{aligned}$$

Substituting $i = 5$ into the IS schedule we can now find the value of Y:

$$\begin{aligned} 0.2Y &= 1100 - 20(5) \\ Y &= 1000/0.2 \\ Y &= 5000 \end{aligned}$$

4.5 Learning outcomes

After working through this chapter and the relevant reading you should be able to:

- State the dimension of a system of equations and be familiar with the terms and concepts of *no solution*, *unique solution*, *multiple solutions* and *infinite solutions* in relation to that system.
- Use graphical methods to solve a $2 * 2$ system of equations.
- Use elimination and substitution to solve a $2 * 2$ system of equations algebraically.
- Identify and sketch simple supply and demand functions.
- Find the equilibrium price and quantity by graphical and algebraic methods.
- Calculate equilibrium income and consumption using the equilibrium equation.
- Analyse IS and LM schedules.

4.6 Sample examination questions

Question 1

a) Solve the equations

$$3x + 5y = 24$$
$$x - 3y = -20$$

[4]

b) A firm has the following supply and demand equations where Q is the quantity and P the price of goods produced.

$$\begin{array}{rrcl} \text{Supply equation} & Q & = & -36 + 4P \\ \text{Demand equation} & Q & = & 174 - 6P \end{array}$$

Find:

i) the value of P which brings equilibrium to the market

ii) the values of P which bring a surplus to the market

iii) the values of P which bring a shortage to the market.

[6]

Question 2

a) Solve the system of equations

$$2x - y = 11$$
$$x + 3y = -5$$

The lines $2x - y = 11$ and $x - 3y = -5$ divide the plane into 4 regions. Which of the two points $(0, 0), (2, -1)$ and $(-2, -2)$ are in the same region?

[6]

b) A company's profits are $P(x) = 20x - 3x^2$ and its costs are $C(x) = 6x + 8$. Sketch these functions for x between 0 and 5 on one graph. Use the graph to give the range of values of x for which the company makes a profit.

[4]

Question 3

a) Solve the system of equations

$$3x - 2y = 8$$
$$-x - 4y = 2$$

[3]

b) Find the break-even point for the monopolistic firm with revenue and cost functions:

$$R(x) = -2x^2 + 14x$$
$$C(x) = 2x + 10$$

[4]

c) Find the level of income Y that brings equilibrium to the economy, given:

$$Y = C + I + G$$
$$C = 300 + 0.75Y$$
$$I = 25 + 0.15Y$$
$$G = 175$$

[3]