Chapter 4

Series and the mathematics of finance

Essential reading

See Chapter 11 of *Dowling* for many further examples of the material covered in this chapter. Answer supplementary problems 11.52 to 11.61 on interest compounding and discounting.

This chapter considers the mathematics of financial calculations. We will study the way that interest is calculated. We will introduce sigma notation and geometric series, and show how these may be used to calculate the future value of a savings plan.

4.1 Simple interest and compound interest

When money is invested, say in a bank or building society, interest is paid so that the amount you have invested keeps up with (or hopefully beats) inflation. Similarly, if you take out a loan then you will be expected to pay interest on the loan — usually at a higher rate than inflation.

Interest payments are usually a percentage of the amount invested or borrowed. There are two ways of calculating these payments - *simple interest* and *compound interest*.

Simple interest

With *simple interest*, the interest received on a sum invested is the same every year.

For example, if $10,000 is invested for three years at a 10% rate of simple interest, then the interest paid would be $1,000 each year. At the end of the three year period (assuming that the original investment and all interest payments have been left in the account) there would be a total of $13,000 in the account.

Compound interest

Simple interest is in fact rarely used. *Compound interest*, which is calculated as a percentage of the total in the account i.e., including interest already received, is used instead.

If \$10,000 is invested for three years at a 10% rate of compound interest, then the interest paid in the first year would be \$1,000. Now there is \$11,000 in the account. So the interest paid in the second year would be 10% of \$11,000 which is \$1,100. In the third year there would be \$12,100 in the account so the interest paid at the end of this year would be \$1,210.

Hence at the end of the three years there would be a total of \$13,310 in the account. This is summarised in the following table.

Year	Amount in Account at Start of Year	Interest Earned	Amount in Account at End of Year
1	\$10,000	\$1,000	\$11,000
2	\$11,000	\$1,100	\$12,100
3	\$12,100	\$1,210	\$13,310

4.1.1 Calculating compound interest

If we are asked to calculate the total amount of compound interest paid after a number of years, then we could work out the interest paid each year and then add these individual payments up to get the total as in the example above.

This is not very efficient however. If the number of years in question is large, say 25 years, then this would be a very time-consuming method. We need to develop a quicker method for calculating compound interest.

Consider the example above again. We have \$10,000 invested for three years at an annual interest rate of 10%. We calculated:

$$final\ \ amount = (((10,000\ .\ 1.1)\ .\ 1.1)\ .\ 1.1)$$

Multiplying the initial investment of \$10,000 by 1.1 to add on 10% at the end of the first year, multiplying this amount by 1.1 and so on for three years.

The calculation above can be simplified to give:

$$final\ \ amount = 10,000\ .\ 1.1^3$$

In general, if a *principal investment* P is invested for t years at an interest rate of r% compounded annually, then the final amount is given by the formula:

$$\boxed{\text{final amount}=P(1+r)^t}$$

Worked example 1

\$1,500 is invested in a bank account which pays interest at a rate of 6% which is compounded annually. If no money is taken out of the bank account, how much will there be in the account at the end of: (i) 5 years, (ii) 8 years, (iii) 15 years?

Solution

Here the principal investment $P = 1500$, the interest rate $r = 6\%$ which means that we multiply the principal investment by 1.06 for each year.

Thus after 5 years, there will be $1500(1.06^5) = \$2007.34$ in the account.

After 8 years, there will be $1500(1.06^8) = \$2390.77$ in the account.

After 15 years, there will be $1500(1.06^{15}) = \$3594.84$ in the account.

Worked example 2

Peter invested a sum of money 10 years ago in an account which paid interest of 4% compounded annually. Peter now has \$555.09 in his account. How much money did Peter originally invest?

Tom invested \$650 into a similar account paying interest of 4% some years ago. The amount that Tom now has in his account is \$855.36. For how many years has Tom's money been invested in the account?

Solution

For Peter we have to solve the equation $555.09 = P \;.\; 1.04^{10}$ to find the principal P which Peter invested.

$$\begin{aligned} 555.09 &= 1.04^{10} \\ P &= \frac{555.09}{1.04^{10}} = 375 \end{aligned}$$

Therefore Peter originally invested \$375.

For Tom we have to solve the equation $855.36 = 650 \;.\; 1.04^t$ to find the number of years t for which Tom has had the account.

$$\begin{aligned} 650 \;.\; 1.04^t &= 855.36 \\ 1.04^t &= \frac{855.36}{650} = 1.3159 \\ t \ln 1.04 &= \ln 1.3159 \\ t &= \frac{\ln 1.3159}{\ln 1.04} = 7 \end{aligned}$$

Therefore Tom has had his money invested in the account for seven years.

Learning activity

1. I invest \$2000 in a bank account which pays interest at a rate of 3.5% compounded annually. How much money will be in the account after:
   (a) 2 years,
   (b) 7 years,
   (c) 20 years?

2. My friend also invests \$2000 for 20 years at an interest rate of 3.2% but her bank pays simple interest. How much more interest than my friend have I earned?
3. Six years ago Anna invested \$8000 in an account that pays interest which is compounded annually. There is now \$10720.77 in Anna's account. What was the rate of interest paid?

4.1.2 Multiple compounding

In all of the examples in the previous section, compound interest was calculated annually. In this section we will consider the case when the interest due is calculated more frequently — maybe every six months, or every month, or even every day.

Suppose \$10,000 is invested for one year at an interest rate of 10%.

If the interest is calculated at the end of the year, then the total amount in the account at the end of the year will be $10,000 \,.\, 1.1 = \$11,000$.

If the interest is compounded every six months, then after one year there will be 2 interest calculations. Interest of $\frac{10}{2} = 5\%$ will be paid twice. We have:

$$final \;\; amount = 10,000 \,.\, 1.05^2 = \$11,025$$

The amount of interest earned increases if the interest is compounded more frequently.

If the interest is compounded every month, then after one year there will be 12 interest calculations. Interest of $\frac{10}{12} = 0.83\%$ will be paid 12 times a year. Starting with the same principal of \$10,000 after one year we would have:

$$final \;\; amount = 10,000(1.0083^{12}) = \$11,047$$

In general, if interest of $r\%$ is awarded m times a year, then starting with a principal amount of P the final amount after t years is given by the following formula:

$$\boxed{\text{final amount} = P(1+\tfrac{r}{m})^{mt}}$$

Learning activity

If a principal of \$10,000 is invested at an interest rate of 10% show that the amount in the account after one year is \$11,051.56 if interest is compounded every day.

4.1.3 Continuous compounding

We have seen that the more frequently interest is compounded the higher the total interest. To push this idea to its limit, we will consider interest which is compounded continuously.

In the previous section we derived the formula:

$$final\ \ amount = P(1+\frac{r}{m})^{mt}$$

where m is the number of times per year that interest is compounded.

If interest is compounded continuously then this formula becomes a *limit* as m tends to infinity.

$$final\ \ amount = P\ lim_{m\to\infty}(1+\frac{r}{m})^{mt}$$

If we replace $\frac{m}{r}$ by n then we can change this formula to:

$$final\ \ amount = P\ \lim_{n\to\infty}(1+\frac{1}{n})^{nrt}$$

Now recall that the definition of the irrational number e is:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Therefore the formula for continual compounding is actually:[1]

$$\boxed{\text{final amount} = Pe^{rt}}$$

[1] You do not need to be able to derive this formula. It should look familiar as it is the same as the growth function formula considered in Chapter 3.

Suppose that our \$10,000 principal is in an account which pays interest of 10% and that the interest is compounded continually. After one year, we will have:

$$final\ \ amount = 10,000(e^{(0.1)(1)}) = \$11,051.71$$

Learning activity

1. \$400 is invested for 2 years in an account which pays interest of 5%. How much interest is earned if:
   (a) interest is compounded annually,
   (b) interest is compounded quarterly,
   (c) interest is compounded continuously?
2. \$750 was invested in an account 5 years ago. The amount in the account is now \$1012.40. Given that the interest was compounded continuously what was the interest rate applied?

4.1.4 Discounting

Sometimes we want to know how much money to invest in order to receive a particular sum in a certain number of years. Calculating the present value of a future sum is known as *discounting*.

Worked example 3

I want to invest some money for my son who is now 7 years old so that he has \$10,000 when he is 18. The savings account I am investing in pays interest of 6% compounded annually. How much money do I need to put into the account now in order to achieve \$10,000 in 11 years time?

There is another account which only pays 5.75% interest but this is compounded continuously. How much would I need to invest in this account to get the same return?

Solution

For the first account where interest is compounded annually we know that

$$final\ \ amount = P(1+r)^t$$

and final amount=\$10,000, $r = 0.06$ and $t = 11$. Substituting these values into the equation and re-arranging we can find the value of the principal amount P needed.

$$\begin{aligned} 10,000 &= P(1.06)^{11} \\ P &= \frac{10,000}{1.06^{11}} \\ P &= 5,267.88 \end{aligned}$$

I would need to invest \$5,267.88 into the account now in order to get a return of \$10,000 in 11 years time.

For the second account where interest is compounded continuously we use the equation

$$final\ \ amount = Pe^{rt}$$

and we have final amount=\$10,000, $= 0.0575$ and $t = 11$ as before. Substituting these values into the equation we can find the value of P.

$$\begin{aligned} 10,000 &= Pe^{(0.0575\ .\ 11)} \\ P &= \frac{10,000}{e^{(0.0575\ .\ 11)}} \\ P &= 5,312.62 \end{aligned}$$

To get a return of \$10,000 from the second account I would need to invest \$5,312.62.

Learning activity

Penny is saving up for a holiday. She wants to save some money so that in three years time she has \$4,000. She is considering putting her money into one of three different accounts. The first account pays interest of 3.8% which is compounded annually. The second account pays interest of 3.8% which is compounded every six months. The third account pays interest of 3.75% which is compounded continuously.

Use discounting to find the principal sum that Penny would need to invest in each different account in order to achieve a return of $4,000 in three years' time. Hence decide which of the accounts Penny should use.

4.2 Geometric series

A *geometric series* is a sum of terms where each term is some multiple of the previous term:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$$

In the geometric series above, the first term is a and all subsequent terms are r times the term before. The number r is called the *geometric ratio* of the geometric series.

Following are some examples of geometric series:

- $2 + 4 + 8 + 16 + 32$ $(a = 2,\ r = 2)$
- $3 + 6 + 12 + 24 + 48$ $(a = 3,\ r = 2)$
- $1 + 5 + 5^2 + 5^3 + \ldots + 5^8$ $(a = 1,\ r = 5)$
- $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ $(a = 2,\ r = \frac{1}{2})$
- $250 + 250(1.06) + 250(1.06)^2 + \ldots + 250(1.06)^{30}$ $(a = 250,\ r = 1.06)$

Learning activity

Decide which of the following are geometric series and for those which are, write down their geometric ratio and the next term.

1. $4 + 8 + 16 + 24 + \ldots$
2. $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$
3. $3 + 9 + 27 + 81 + 243 + 729 + \ldots$
4. $100 + 75 + 56.25 + 42.1875 + \ldots$
5. $0.1 + 0.01 + 0.001 + 0.0001 + \ldots$
6. $427 + 427(2) + 427(3) + 427(4) + 427(5) + 427(6) + \ldots$

It is easy to calculate the sum of a geometric series if just a few terms are involved. Simply add the terms together. However this becomes much harder if there are many terms in the series. We are going to derive a formula which can be used to calculate the sum of a geometric series. [2]

[2] You do not need to be able to derive this formula yourself - however it is often easier to remember a formula if you understand why it works.

Consider the geometric series with n terms and sum S_n:

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

Multiplying every term by r gives us:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$$

Now if we subtract S_n from rS_n all of the terms will cancel except for ar^n and a. Thus we are left with:

$$rS_n - S_n = ar^n - a$$

Now rearranging to make S_n the subject of the formula:

$$S_n(r-1) = a(r^n - 1)$$

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

For example, consider the geometric series $100 + 50 + 25 + 12.5 + \ldots$ which has $a = 100$ and $r = 0.5$.

The sum of the first five terms $S_5 = 100 + 50 + 25 + 12.5 + 6.25 = 193.75$. Using the formula derived above we have:

$$S_5 = 100\frac{(0.5^5 - 1}{0.5 - 1} = 100\frac{-0.96875}{-0.5} = 193.75$$

The sum of the first 10 terms is given by:

$$S_{10} = 100\frac{0.5^{10} - 1}{0.5 - 1} = 100\frac{-0.999023}{-0.5} = 199.98\ (2d.p.)$$

Learning activity

Use the formula above to calculate the sum of the following geometric series to the number of terms stated.

1. $2 + 6 + 18 + 54 + \ldots$ (8 terms)
2. $2 + 10 + 50 + 250 + \ldots$ (12 terms)
3. $1 + 3 + 9 + 27 + \ldots$ (20 terms)
4. $8 + 4 + 2 + 1 + \frac{1}{2} + \ldots$ (10 terms)
5. $8 - 4 + 2 - 1 + \frac{1}{2} + \ldots$ (10 terms)

4.2.1 Geometric series and finance

Geometric series can be used to analyse savings and loans. In the case of saving we consider the situation where a regular sum of money is invested at the same time each year. For example, suppose I invest \$1,000 at the start of each year into an account which pays interest of 6%. If I make these payments each year for four years, then how much money do I have in the account at the end of four years?

The first payment of \$1000 is invested for four years and so its future value is $1000(1.06)^4 = \$1262.48$.

The second payment of \$1000 is invested for three years and its future value is $1000(1.06)^3 = \$1191.02$.

The third payment is invested for two years and its future value is $1000(1.06)^2 = \$1123.60$.

The fourth and final payment is only invested for one year and its value at the end of this year is $1000(1.06) = \$1060$.

Therefore at the end of the four years I would have a total of:

$$1262.48 + 1191.02 + 1123.60 + 1060.00 = \$4637.10$$

The sum that we have calculated is actually:

$$1000(1.06) + 1000(1.06)^2 + 1000(1.06)^3 + 1000(1.06)^4$$

Writing it like this, we can see that it is in fact a geometric series with $a = 1000(1.06) = 1060$ and $r = 1.06$. We can use the formula to find the sum in the account after four years:

$$S_4 = \frac{1060(1.06^4 - 1)}{1.06 - 1} = 4637.09$$

[3]

We can also calculate the amount that would be in the account if I continued to invest \$1000 per year at the same rate of interest for 20 years:

$$S_{20} = \frac{1060(1.06^{20} - 1)}{1.06 - 1} = \$38,992.73$$

And if I managed to save for 50 years I would have over \$300,000.

$$S_{50} = \frac{1060(1.06^{50} - 1)}{1.06 - 1} = \$307,756.06$$

[3] The slight discrepancy in the answers here is due to rounding errors.

Learning activity

1. The sum of \$4200 is invested annually at 5% interest per annum. What is the total sum of money in the account at the end of 50 years?
2. A regular saving of \$500 is made at the start of every year for 10 years. Determine the value of the savings at the end of the tenth year on the assumption that the rate of interest is:
   (a) 11% compounded annually
   (b) 10% compounded continuously. (Write down the first terms of the geometric series to find the value of a and r).
3. A sum of \$1000 is invested annually at 7.5% interest per annum.
   (a) What is the total sum of money at the end of n years?
   (b) What is the total sum of money at the end of 20 years?
4. The formula $S_n = \frac{a(r^n-1)}{r-1}$ does not apply for $r = 1$. What is the correct formula for S_n in the case when $r = 1$?

4.3 Sigma notation

Sigma notation is a useful mathematical shorthand for describing the sum of a sequence of numbers or terms. *Sigma* is the name of the Greek letter Σ.

$$\sum_{i=a}^{b} u_i$$

means the sum of all u_i evaluated for all i between a and b.

For example,

$$\sum_{i=1}^{6} 5^i = 5^1 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6$$

$$\sum_{i=0}^{48} 2i + 1 = 1 + 3 + 5 + \ldots + 95 + 97$$

$$\sum_{i=5}^{8} \frac{i-1}{i} = \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8}$$

To write a given series in Σ form, we have to identify the *general term* u_i and the starting and end points for the *counter* i.

For example, consider the series:

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 20$$

This series can be rewritten as:

$$(1\ .\ 2) + (2\ .\ 2) + (3\ .\ 2) + \ldots + (10\ .\ 2)$$

Therefore the *general term* is $(i\ .\ 2)$ or $2i$ where i runs from $i = 1$ to $i = 10$. So the series can be written in Σ notation as:

$$\sum_{i=1}^{10} 2i$$

Learning activity

1. Write out the terms of the following series:
   (a)
   $$\sum_{i=1}^{5} \frac{1}{i}$$
   (b)
   $$\sum_{i=3}^{7} i^2$$

(c)

$$\sum_{i-1}^{5} \frac{1}{i(i+1)}$$

(d)

$$\sum_{i=1}^{6} e^i$$

(e)

$$\sum_{i=4}^{8} (-1)^i(2i+3)$$

(f)

$$\sum_{i=0}^{5} (i+1)^3$$

2. Rewrite using Σ notation:
   (a) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{9}$
   (b) $2^2 + 4^2 + 6^2 + \ldots + 100^2$
   (c) $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{99}$
   (d) $1^3 - 2^3 + 3^3 - 4^3 + \ldots + 19^3$
   (e) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots + \frac{99}{100}$
3. Use Σ notation to write:
   (a) The sum of the first 60 odd numbers.
   (b) The sum of all the square numbers from 100 to 400 inclusive.
   (c) The sum of all the numbers between 1 and 100 inclusive that leave a remainder 1 when divided by 7.

4.4 Learning outcomes

After studying this chapter and the relevant reading you should be able to:

- Explain the difference between simple interest and compound interest.
- Calculate the future value of a principal under annual compounding, multiple compounding and continuous compounding.
- Use discounting to calculate the present value of a future sum.
- Recognise a geometric series and find the geometric ratio of such a series.
- Sum a geometric series using the formula $S_n = \frac{a(r^n-1)}{r-1}$
- Solve problems involving savings and loans which can be formulated using geometric series.
- Write out the terms of a series given in Σ notation.
- Identify the general term and number of terms of a given series and hence write the series in Σ notation.

4.5 Sample examination questions

Question 1

a) Write down the formula for the sum of a geometric progression of n terms which has the first term equal to a and geometric ratio r. Using the formula and showing all your working evaluate:

i)

$$\sum_{k=-2}^{8} 5 \, . \, 2^k$$

ii)

$$\sum_{k=1}^{5}(3^k + 2^{k-1})$$

[5]

b) I invest P dollars in a bank for n years and receive interest at $r\%$ per annum, compounded annually. Express mathematically the value of my investment after n years. Calculate the value of the investment when $P = 2,000$, $n = 5$ and $r = 6$.

[2]

c) My friend invested $1,000 for 10 years at a different interest rate, again compounded annually. At the end of the 10 years her investment was worth $1,628.89. What was the rate of interest?

[3]

Question 2

a) Determine the value of

i)

$$\sum_{r=1}^{5} r(r-1)$$

ii)

$$\sum_{r=-2}^{2} 2^r$$

[3]

b) Rewrite the following sums using the *sigma* notation:

i) $1 + 4 + 9 + 16 + ... + 100$;

ii) $25 + 22 + 19 + 16 + ... + 1 + (-2) + (-5)$.

[3]

c) Mary invests A dollars at the beginning of *every* year in a bank which offers fixed rate interest of $r\%$ compounded annually. Use the *sigma* notation to express the total amount of Mary's investment after n years.

[4]